

Differential Galois Groups and Representations of Quivers for Seismic Models with Constant Hessian of Square of Slowness[†]

Primitivo Acosta-Humánez, Hernán Giraldo, and Carlos Piedrahita*

ABSTRACT. The trajectory of energy is modeled by the solution of the Eikonal equation, which can be solved by solving a Hamiltonian system. This system is amenable of treatment from the point view of the theory of Differential Algebra. In particular, by Morales-Ramis theory it is possible to analyze integrable Hamiltonian systems through the abelian structure of their variational equations. In this paper we obtain the abelian differential Galois groups and the representation of the quiver, that allow us to obtain such abelian differential Galois groups, for some seismic models with constant Hessian of square of slowness, proposed in [20], which are equivalent to linear Hamiltonian systems with three uncoupled harmonic oscillators.

Keywords and Phrases. Differential Galois Theory, Eikonal Equation, Hamilton Equation, Helmholtz Equation, High Frequency Approximation, Morales-Ramis Theory, Ray Theory, Representations of Quivers

MSC 2010. 12H05, 16G20, 34E05, 34E20, 35C20, 35Q60, 35L05, 86A15

1. Introduction

Wave equations appear in many areas of sciences, in particular, in seismology and seismic exploration. Asymptotic solutions give us an estimation to the actual solution to cases where it is difficult to obtain exact solutions. In this context, the idea of Ray theory is a very important tool used in geophysics. High frequency approximations or ray theory have been used in seismology to obtain good approximation for the solution of the acoustic and/or elastic wave partial differential equations (PDE). They represent the transmission of energy in a continuous media. The acoustic wave equation and Transport equation are second order linear PDE. The first one is transformed into the *Eikonal equation*, which is a first order nonlinear PDE. The trajectory of energy is modeled by the solution of the Eikonal Equation and it can be solved by solving a Hamiltonian system. For more details, see [18].

[†] Supported by COLCIENCIAS-ECOPETROL project (Contract RC. No. 0266-2013) and CODI (Estrategia de Sostenibilidad 2016-2017, Universidad de Antioquia UdeA)

* Corresponding author

Galois theory in the context of linear differential equations is known as *differential Galois theory* or also as *Picard-Vessiot theory*, see [27, 30, 31, 40, 41]. There are effective algorithms and procedures derived from differential Galois theory such as Kovacic Algorithm (see [29]), Kimura's Table (see [28]) and *Algebrization Procedure* (see [6, 10–12]) that allow us to obtain explicit solutions, whenever it can be possible, of differential equations. Differential Galois theory has been applied successfully in mathematical physics. One of these applications corresponds to Morales-Ramis theory (see [32]), that is, the differential Galois theory linked with the non-integrability of dynamical systems, being the starting point to prove non-integrability of Hamiltonian and non-Hamiltonian systems, see [1–5, 11]. Similarly, differential Galois theory has been used to study integrability and non-integrability of polynomial vector fields (see [7, 14]), integrability in quantum mechanics (see [6–10, 12]) and integrability in quantum optics (see [13]).

In the sixties and seventies P. Gabriel, in [24], showed a correspondence between algebras of finite dimension over algebraically closed fields with quivers. After that, started a great development of the theory of representation of finite dimensional algebras and representation of quivers. References for this are [17], [16], [25], [26], [37] and [38].

The connection between the representations of quivers (directed graphs) and differential Galois group is that both try to find the solutions to systems of differential equations. Representation of quivers is used with great success in the theory of representations of finite dimensional algebras. In particular, using representations we can solve a system of m differential equations of degree s in n indeterminates (see [21], Section 1). Moreover, we can see the representation theory of algebras as a generalization of linear algebra, in the sense that in linear algebra we have a single linear transformation acting on a vector space, while in representation theory we have simultaneously a finite number of linear transformations acting on a vector space.

In this paper we study some seismic problems with a constant Hessian of square of slowness without interactions between positions by using Morales-Ramis theory and representation theory of quivers. Firstly, we write the equivalent Hamiltonian to the Eikonal equation with constant Hessian of the square of slowness without interactions between the positions, which is equivalent to study a linear Hamiltonian system of three uncoupled oscillators. After, due to the linearisation of the Hamiltonian vector field, the so-called variational equation, is equivalent to the linear Hamiltonian system. Then we compute the differential Galois groups and the representation of the quiver, that allow us to obtain such abelian differential Galois groups, for such systems.

2. Preliminaries

In this section, which is divided in three parts, we give the basic theoretical background to understand this article. In the first part we present a brief review of ray theory applied to seismic exploration. The second part contains a brief summary of differential Galois theory, which is Galois theory in the context of linear differential equations, including the Morales-Ramis theory to study the integrability

of Hamiltonian Systems. The third part is devoted to an elementary background about theory of representations of quivers, which will be related with these Ray equations and differential Galois theory to explore new relations and obtain results that have a mathematical interest.

2.1. Ray theory. Ray theory is a collection of theorems that have been developed in the field of asymptotic analysis. It is very useful as an approximation to the solution of ODEs or PDEs, in the context of the so called high-frequency approximation. Essentially, we suppose an asymptotic expansion of the unknown solution, see [18, 22, 42], in the frequency domain as

$$(2.1) \quad \begin{aligned} u(\mathbf{x}, \omega) &\sim \omega^\beta \exp^{i\omega\tau(\mathbf{x})} \sum_{j=0}^{\infty} \frac{A_j(\mathbf{x})}{(i\omega)^j} \\ \tau(\mathbf{x}) &\equiv \text{Transport time,} \\ A_j(\mathbf{x}) &\equiv \text{Independent width of frequency,} \\ j &= 0, 1, 2, \dots \end{aligned}$$

The physical interpretation of the series (2.1) will be analyzed: the inverse Fourier transform

$$(2.2) \quad \omega^\beta \exp^{i\omega\tau(\mathbf{x})} \rightarrow F(t - \tau(\mathbf{x})),$$

is the frequency transform of a progressive wave, in which $F(t)$, is the inverse transform of ω^β , which propagates in the direction of increment of $\tau(\mathbf{x})$.

In the series (2.1), the terms that are divided by higher powers of $(i\omega)$ represents successive integrations in the time domain, which can be interpreted as the smoothest terms of the original function, F . In the time domain, we can interpret this as an infinite chain of distributions each more smooth than the one before, added together. For example, consider the chain of distributions

$$\delta(t) \rightarrow H(t) \rightarrow r(t),$$

where $\delta(t)$, represents the delta function, $H(t)$, represents the Heaviside distribution and $r(t)$, represents the ramp function. The Fourier transform of the first term represents the portion more singular of the solution, associated with the high frequencies, while the subsequent terms, represent the smoothest contributions, i.e., the low frequency terms of the solution of the Helmholtz's Equation

$$(2.3) \quad LP = \left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{x})} \right] P(\mathbf{x}, \omega) = 0.$$

Considering the wave equation in the frequency domain, i.e., the Helmholtz Equation, we substitute the asymptotic series. A formal definition of an asymptotic expansion can be found in [19]. Substituting this series in the Helmholtz Equation we obtain:

$$(2.4) \quad \begin{aligned} LP = & \omega^\beta \exp i\omega\tau \sum_{j=0}^{\infty} \frac{1}{(i\omega)^j} \left[\omega^2 \left\{ \frac{1}{c^2} - (\nabla)^2 \right\} A_j \right. \\ & \left. + i\omega \{ 2\nabla\tau \cdot \nabla A_j + A_j \nabla^2 \tau \} + \nabla^2 A_j \right]. \end{aligned}$$

The equation (2.4) is satisfied whether the coefficients of the series to be equal to zero, see [20], therefore equating to zero each power of $i\omega$, starting with the maximum power, $\beta + 2$, when $j = 0$, we obtain the so called *Eikonal Equation*

$$(2.5) \quad (\nabla\tau(\mathbf{x}))^2 - \frac{1}{c^2(\mathbf{x})} = 0.$$

For the series (2.4), the highest non-zero power of ω is $\beta + 1$, which is obtained when $j = 0$, and thus we obtain the so called Transport Equation of order zero

$$(2.6) \quad 2\nabla\tau(\mathbf{x}) \cdot \nabla A_0(\mathbf{x}) + A_0 \nabla^2 \tau(\mathbf{x}) = 0.$$

The Eikonal Equation represents the travel time and geometrically, represents the wave fronts of the wave. If we consider the Eikonal Equation, we can solve it by the method of characteristics, see [23]. In this way, the first order non-linear partial differential equation is converted to a system of ordinary differential equations, which corresponds to a Hamiltonian system, see [15].

Consequently, we obtain trajectories defined by the following system, see [18, 20, 36]:

$$(2.7) \quad \begin{aligned} \frac{d\mathbf{x}}{d\sigma} &= \mathbf{p} \\ \frac{d\mathbf{p}}{d\sigma} &= \frac{1}{2} \nabla \left(\frac{1}{c^2(\mathbf{x})} \right) = \frac{-\nabla c(\mathbf{x})}{c^3(\mathbf{x})} \\ \frac{d\tau}{d\sigma} &= \frac{1}{c^2(\mathbf{x})}. \end{aligned}$$

These trajectories represent the flow of energy. The physical interpretation of the variables of the system (2.7) is the following, see [19]: x_i , represents the coordinates along the ray and the p_i represents the coordinate of the so called slowness vector. This vector has units of the inverse of the velocity of the propagation of the wave in the media. The variable τ represents the travel time along the ray and σ represents an integration parameter, which can be selected in different ways according to its convenience. The physical ray is the vector $x_i, i = 1, 2, 3$. Finally, the function $c(\mathbf{x})$ represents the velocity field. It is a smooth function that represents the variations of the wave propagation along a medium, and it is a scalar function defined in a domain of \mathbb{R}^3 , which represents macroscopically the structure of the medium. The scalar field $c^{-1}(\mathbf{x})$ is called the *slowness* and $c^{-2}(\mathbf{x})$ is called the *square of slowness*, see [20].

The ray is therefore a path in space, where energy is propagated between a source and a receiver, and it is a good approximation when the length of the wave

is small compared to the object of interest. It is usually called the *high frequency approximation*, see [39].

The system just described, is the one to which we will apply the concepts of differential algebra to analyze the integrability of the ODE system that appears in seismology.

In particular, considering the 2D subcase corresponding to the plane x_1x_3 , that is, setting $x_2 = 0$ and for instance $\mathbf{x} = (x_1, 0, x_3)$, the acoustic wave equation is given by

$$\nabla^2 P = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2},$$

which can be written as

$$\frac{\partial^2 P}{\partial x_1^2} + \frac{\partial^2 P}{\partial x_3^2} = \frac{1}{c^2(\mathbf{x}, t)} \frac{\partial^2 P}{\partial t^2},$$

being $P(\mathbf{x}, t)$ the pressure in the high frequency approximation. Thus,

$$P(\mathbf{x}, t) = A(\mathbf{x}, t) e^{-i\tau(\mathbf{x}, t)}, \quad w \gg 1, \quad (\text{equivalently } \lambda \ll 1).$$

Therefore the Eikonal equation (2.5) in this case can be written as

$$(2.8) \quad \left(\frac{\partial \tau}{\partial q_1} \right)^2 + \left(\frac{\partial \tau}{\partial q_3} \right)^2 = \frac{1}{c^2(\mathbf{q})}, \quad p_1^2 + p_3^2 - \frac{1}{c^2} = 0,$$

where $\mathbf{q} = (q_1, q_2, q_3)$, $q_i = x_i$, $p_i = \frac{\partial \tau}{\partial q_i}$ and $c(\mathbf{q})$ is the velocity field, being $q_2 = p_2 = 0$. In the context of Hamiltonian mechanics the vector \mathbf{q} represents the generalized coordinates and the vector \mathbf{p} represents the generalized momentum, see [15]. We recall that the Hamiltonian corresponding to the Eikonal equation (2.8) has two degrees of freedom and it is a particular case of a Hamiltonian with three degrees of freedom ($q_2, p_2 \in \mathbb{C}$). Moreover, we will study the integrability of quadratic Hamiltonian with three degrees of freedom without interaction of the positions, which are equivalent to seismic models with constant Hessian of square of slowness.

Finally, in a quadratic Hamiltonian with three degrees of freedom the only possible interactions of the positions are given by $q_1q_2, q_2q_1, q_1q_3, q_3q_1, q_2q_3$ and q_3q_2 . Although the Hessian of quadratic hamiltonians with interaction in the positions is constant, we will not consider this case in this paper.

2.2. Differential Galois and Morales-Ramis Theories. Picard-Vessiot theory is the Galois theory of linear differential equations, which is also known as differential Galois theory. Although we will review here some of its main definitions and results, we refer the reader to [41] for a wide theoretical background. See also [10, 12] for short summaries of this theory.

By $M_n(k)$ we means the set of $n \times n$ square matrices with entries in the field k . We recall that the general linear group over \mathbb{C} is given by

$$\mathrm{GL}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det(A) \neq 0\}.$$

Therefore, an algebraic group of matrices 2×2 is a subgroup $G \subset \mathrm{GL}_2(\mathbb{C}) \subset M_2(\mathbb{C})$ defined by means of algebraic equations in its matrix elements and in the inverse

of its determinant. That is, there exists a set of polynomials $P_i \in \mathbb{C}[x_1, \dots, x_5]$, for $i \in I \subseteq \mathbb{N}$, such that $A \in \mathrm{GL}_2(\mathbb{C})$ given by

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

belongs to G if and only if $P_i(x_1, x_2, x_3, x_4, x_5) = 0$ for all $i \in I \subseteq \mathbb{N}$ and where $x_5 = (\det A)^{-1}$. It is said that G is an algebraic manifold endowed with a group structure. Recall that $A \in \mathrm{SL}_2(\mathbb{C})$ iff $A \in \mathrm{GL}_2(\mathbb{C})$ and $\det A = 1$.

A group G is called *solvable* if and only if there exists a chain of normal subgroups

$$e = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G,$$

satisfying that the quotient G_i/G_j is abelian for all $n \geq i \geq j \geq 0$.

It is well known that any algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. In particular, the *identity connected component* G^0 of G is defined as the largest connected algebraic subgroup of G containing the identity. In case that $G = G^0$ we say that G is a *connected group*. Moreover, if G^0 is solvable we say that G is *virtually solvable*.

The following result provides the relationship between virtual solvability of an algebraic group and its structure.

THEOREM 2.1 (Lie-Kolchin). *Let $G \subseteq \mathrm{GL}_2(\mathbb{C})$ be a virtually solvable group. Then, G^0 is triangularizable, i.e., it is conjugate to a subgroup of upper triangular matrices.*

Now, we briefly introduce Picard-Vessiot Theory. First, we say that $(\mathcal{K}, ')$ - or, simply, \mathcal{K} - is a *differential field* if \mathcal{K} is a commutative field of characteristic zero, depending on x and $'$ is a derivation on \mathcal{K} (that is, satisfying that $(a+b)' = a' + b'$ and $(a \cdot b)' = a' \cdot b + a \cdot b'$ for all $a, b \in \mathcal{K}$). We denote by k the *field of constants of \mathcal{K}* , defined as $k = \{c \in \mathcal{K} \mid c' = 0\}$.

We deal with second order linear homogeneous differential equations, i.e., equations of the form

$$(2.9) \quad y'' + b_1 y' + b_0 y = 0, \quad b_1, b_0 \in \mathcal{K},$$

and we are concerned with the algebraic structure of their solutions. Moreover, along this work, we refer to the current differential field as the smallest one containing the field of coefficients of this differential equation.

Let us suppose that y_1, y_2 form a basis of solutions for the Equation (2.9), i.e., y_1, y_2 are linearly independent over k and every other solution is a k -linear combination of y_1 and y_2 . Let $\mathcal{L} = \mathcal{K}\langle y_1, y_2 \rangle = \mathcal{K}(y_1, y_2, y_1', y_2')$ be the differential extension of \mathcal{K} such that k is the field of constants for both \mathcal{K} and \mathcal{L} . In this terms, we say that \mathcal{L} , the smallest differential field containing \mathcal{K} and $\{y_1, y_2\}$, is the *Picard-Vessiot extension* of \mathcal{K} for the differential equation given in Equation (2.9).

The group of all the differential automorphisms of \mathcal{L} over \mathcal{K} that commute with the derivation $'$ is called the *differential Galois group* of \mathcal{L} over \mathcal{K} , which is denoted by

$\mathrm{DGal}(\mathcal{L}/\mathcal{K})$. This means in particular that for any $\sigma \in \mathrm{DGal}(\mathcal{L}/\mathcal{K})$, $\sigma(a') = (\sigma(a))'$ for all $a \in \mathcal{L}$ and that $\sigma(a) = a$ for all $a \in \mathcal{K}$. Thus, if $\{y_1, y_2\}$ is a fundamental system of solutions of Equation (2.9) and $\sigma \in \mathrm{DGal}(\mathcal{L}/\mathcal{K})$ then $\{\sigma y_1, \sigma y_2\}$ is also a fundamental system. This implies the existence of a non-singular constant matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

This fact can be extended in a natural way to a system

$$\sigma \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(y'_1) & \sigma(y'_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} A_\sigma,$$

which leads to a faithful representation $\mathrm{DGal}(\mathcal{L}/\mathcal{K}) \rightarrow \mathrm{GL}_2(\mathbb{C})$ and makes feasible to consider $\mathrm{DGal}(\mathcal{L}/\mathcal{K})$ as a subgroup of $\mathrm{GL}_2(\mathbb{C})$ depending (up to conjugacy) on the choice of the fundamental system $\{y_1, y_2\}$.

One of the fundamental results of the Picard-Vessiot Theory is the following theorem (see [27]).

THEOREM 2.2. *The Galois group $\mathrm{DGal}(\mathcal{L}/\mathcal{K})$ is an algebraic subgroup of $\mathrm{GL}_2(\mathbb{C})$.*

We say that the Equation (2.9) is *integrable* if the Picard-Vessiot extension $\mathcal{L} \supset \mathcal{K}$ is obtained as a tower of differential fields $\mathcal{K} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L}$ such that $\mathcal{L}_i = \mathcal{L}_{i-1}(\eta)$ for $i = 1, \dots, m$, where either

- (i) η is *algebraic* over \mathcal{L}_{i-1} , that is η satisfies a polynomial equation with coefficients in \mathcal{L}_{i-1} ;
- (ii) η is *primitive* over \mathcal{L}_{i-1} , that is $\eta' \in \mathcal{L}_{i-1}$; and
- (iii) η is *exponential* over \mathcal{L}_{i-1} , that is $\eta'/\eta \in \mathcal{L}_{i-1}$.

Usually in terms of differential algebra's terminology, we say that the Equation (2.9) is integrable if the corresponding Picard-Vessiot extension is *Liouvillian*. Moreover, the following theorem holds.

THEOREM 2.3 (Kolchin). *The Equation (2.9) is integrable if and only if $\mathrm{DGal}(\mathcal{L}/\mathcal{K})$ is virtually solvable, that is, its identity component $(\mathrm{DGal}(\mathcal{L}/\mathcal{K}))^0$ is solvable.*

For instance, for the case $b_1 = 0$ in the Equation (2.9), i.e. $y'' + b_0 y = 0$, it is very well known that $\mathrm{DGal}(\mathcal{L}/\mathcal{K})$ is an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$, see [27, 41]. In particular, \mathbb{G}_a (the additive group) and \mathbb{G}_m (the multiplicative group) are algebraic subgroups of $\mathrm{SL}(2, \mathbb{C})$, which are given by

$$\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \in \mathbb{C} \right\}, \quad \mathbb{G}_m = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{C}^* \right\}.$$

We recall that $x = x_0 \in \mathbb{C}$ is an ordinary point of the Equation (2.9) whether b_0 and b_1 are analytics in x_0 . On the other hand, we say that x_0 is a singular point of the Equation (2.9) whether x_0 is not an ordinary point for (2.9). Moreover, if x_0 is a singular point of the Equation (2.9), we say that it is regular provided that $(x - x_0)^2 b_0$ and $(x - x_0) b_1$ are analytic functions of the Equation (2.9). In case that a point is neither ordinary nor singular regular, we say that such point is singular irregular. To study $x = \infty$ as an ordinary point or singular (regular or

irregular) point of the Equation (2.9), we make the change of variable $x = z^{-1}$ and then we study the behaviour of $z = 0$ in the new equation. This process is a particular case of the *Hamiltonian Algebrization Procedure*. For further details see [4, 10].

One important application of differential Galois theory is *Morales Ramis Theory*. Let us consider a n degrees of freedom Hamiltonian H , given by

$$(2.10) \quad H = \frac{\|\mathbf{p}\|^2}{2} + V(\mathbf{q}), \quad \mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \mathbf{q} \in \mathbb{C}^n, \quad \mathbf{p} \in \mathbb{C}^n.$$

The equations of the flow of the Hamiltonian system, in a system of canonical coordinates (\mathbf{q}, \mathbf{p}) , are given by

$$(2.11) \quad \dot{z} = X_H, \quad z = (\mathbf{q}, \mathbf{p}), \quad X_H = J_n \nabla H, \quad J_n = \begin{pmatrix} \mathbf{0}_n & I_n \\ -I_n & \mathbf{0}_n \end{pmatrix},$$

where $\mathbf{0}_n$ is the square null matrix of size n^2 , I_n is the identity matrix of size n^2 , the matrix J_n is known as the *symplectic matrix*, and the vector field X_H is known as the *Hamiltonian vector field*. The system (2.11) is known as the *Hamilton equations*, which are conventionally written as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n.$$

For instance the Hamiltonian vector field is then written as

$$X_H = \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ -\frac{\partial V(\mathbf{q})}{\partial q_1} \\ \vdots \\ -\frac{\partial V(\mathbf{q})}{\partial q_n} \end{pmatrix}.$$

Assume that $\gamma = \gamma(t) = (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t))$ is a particular integral curve (also known as particular solution) of the Hamiltonian system (2.11). The Jacobian of the Hamiltonian vector field X_H restricted to γ , which is exactly the product between the symplectic matrix and the Hessian of the Hamiltonian H , is the matrix of coefficients for the *variational equation* of the Hamiltonian system (2.11), which is given by

$$(2.12) \quad \dot{\eta} = A(t)\eta, \quad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{2n} \end{pmatrix}, \quad A(t) = X'_H(\gamma) = J \text{Hess}(H)|_\gamma.$$

The matrix $A(t)$ can be written explicitly as

$$A(t) \begin{pmatrix} \mathbf{0}_n & I_n \\ -B_n & \mathbf{0}_n \end{pmatrix}, \quad B_n = \begin{pmatrix} \frac{\partial^2 V(\mathbf{q})}{\partial q_1^2} & \dots & \frac{\partial^2 V(\mathbf{q})}{\partial q_n \partial q_1} \\ \vdots & & \\ \frac{\partial^2 V(\mathbf{q})}{\partial q_1 \partial q_n} & \dots & \frac{\partial^2 V(\mathbf{q})}{\partial q_n^2} \end{pmatrix}.$$

We recall that the *Poisson bracket* between $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$ is given by

$$\{f, g\} = \sum_{k=1}^n \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} \right).$$

We say that f and g are in *involution* when $\{f, g\} = 0$, also we say in this case that f and g commute under the Poisson bracket.

A Hamiltonian H in \mathbb{C}^{2n} is called *integrable in the sense of Liouville* provided that there exist n independent first integrals of the Hamiltonian system in involution. Following [32–35], we say that H is *integrable by terms of rational functions* if we can find a complete set of integrals within the family of rational functions. Respectively, we can say that H is *integrable by terms of meromorphic functions* if we can find a complete set of integrals within the family of meromorphic functions. *Morales-Ramis theory* relates integrability of Hamiltonian systems in the Liouville sense with integrability Picard-Vessiot theory in terms of differential Galois theory. The following theorem is known as *Morales-Ramis theorem*, see [32–35].

THEOREM 2.4 (Morales-Ramis). *Let H be a Hamiltonian in \mathbb{C}^{2n} , and γ be a particular solution such that the variational has regular (resp. irregular) singularities at the points of γ at infinity. If H is integrable by terms of meromorphic (resp. rational) functions, then the differential Galois group of the variational equation is virtually abelian.*

This theorem has been applied to prove non-integrability in some physical problems, see [1, 2, 4, 11] and references therein. We recall that any linear Hamiltonian system is integrable and its corresponding Hamiltonian has the form

$$(2.13) \quad H = \frac{\|\mathbf{p}\|^2}{2} + \mathbf{q}^T M \mathbf{q},$$

where M is a n^2 size matrix with entries in \mathbb{C} . In particular, we are interested in the case where M is a diagonal matrix, i.e., our main object is a Hamiltonian with uncoupled harmonic oscillators.

2.3. Representations of quivers. A *quiver* Γ is a quadruple $\Gamma = (\Gamma_0, \Gamma_1, s, e)$, where Γ_0 and Γ_1 are sets, Γ_0 is called *the set of vertices* and Γ_1 *the set of arrows* (between vertices) and $s, e : \Gamma_1 \rightarrow \Gamma_0$ are maps given by $s(\alpha) = i$ and $e(\alpha) = j$ for all arrow $\alpha : i \rightarrow j$, that is, α is an arrow from the vertex i to the vertex j . A *path* in Γ is a sequence of arrows $p = \alpha_n \cdots \alpha_1$ such that $e(\alpha_t) = s(\alpha_{t+1})$ for $1 \leq t \leq n-1$ (or $p_1 = (j|\alpha_n \cdots \alpha_1|i)$ to mean that p_1 is a path from the vertex i to the vertex j). We denote $l(p) = n$ the *length* of the arrow p . A *path of length zero* (or *trivial path*) is a path without arrows and for each vertex $i \in \Gamma_0$, we associate a trivial path denoted by ϵ_i , such that $s(\epsilon_i) = e(\epsilon_i) = i$.

Let k be a field and Γ a *finite quiver*, that is, the sets Γ_0 and Γ_1 are finite. For k and Γ , we define an algebra, which we will denote by $k\Gamma$. Let B be the set of all paths in Γ , and we take $k\Gamma$ the k -vector space with basis B . Take two paths in B , namely $p_1 = (j|\alpha_n \cdots \alpha_1|i)$ and $p_2 = (r|\beta_m \cdots \beta_1|l)$. We define the multiplication of p_1 and p_2 as follow:

$$p_1 \cdot p_2 = \begin{cases} (j|\alpha_n \cdots \alpha_1 \beta_m \cdots \beta_1|l), & \text{whether } i = r \\ 0 & \text{otherwise} \end{cases}.$$

The multiplication in $k\Gamma$ is extended linearly. The k -algebra $k\Gamma$ is called *the path algebra* of Γ . We denote by $f.d.(k\Gamma)$ the category of finitely generated modules over the path algebra $k\Gamma$ and by J the ideal of $k\Gamma$ generated by all arrows of Γ . An ideal I of $k\Gamma$ is *admissible* if there exists $n > 0$ such that $J^n \subset I \subset J^2$.

It is well-known that many algebras can be expressed as path algebras. More precisely, if A is an indecomposable finite basic algebra over an algebraically closed field k , then there is a quiver Γ_A and an epimorphism of algebras $\phi : k\Gamma_A \rightarrow A$ such that $\ker\phi$ is an admissible ideal of $k\Gamma$ (Gabriel's Theorem).

Given a field k and a finite quiver Γ , we can associate representations and the corresponding morphisms to obtain the so called representation category of Γ , usually denoted by $rep\Gamma$. It is well-known that if k is a field and Γ is a finite quiver, then $rep\Gamma$ and $f.d.(k\Gamma)$ are equivalent categories.

Now we describe the representations for the quiver

$$\Gamma^n : \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \dots \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \quad (n \geq 1),$$

where the edges can be changed by arrows in any direction. If $p = \alpha_n \cdots \alpha_1 \in \Gamma^n$ is a path of length $n \geq 1$, then there exists an indecomposable finite generated $k\Gamma$ -module $M[p]$, which can be described as follows. There is an ordered k -basis $\{z_0, z_1, \dots, z_n\}$ of $M[p]$ such that the action of $k\Gamma$ on $M[p]$ is given by the following representation

$$\varphi_p : k\Gamma \rightarrow M_{n+1}(k).$$

Let $\mathbf{v}(j) = \mathbf{t}(\alpha_{j+1})$ for $0 \leq j \leq n-1$ and $\mathbf{v}(n) = \mathbf{s}(\alpha_n)$. Then for each vertex $i \in \Gamma_0^n$ and for each arrow $\gamma \in \Gamma_1^n$ and for all $0 \leq j \leq n$ define

$$\varphi_p(i)(z_j) = \begin{cases} z_j, & \text{if } \mathbf{v}(j) = i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_p(\gamma)(z_j) = \begin{cases} z_{j-1}, & \text{if } \alpha_j = \gamma \\ 0, & \text{otherwise} \end{cases}$$

We call φ_p the *canonical representation* and $\{z_0, z_1, \dots, z_n\}$ a *canonical k -basis* for $M[p]$. In particular, in this paper we consider $k = \mathbb{C}$.

3. Main results

The results presented in this paper are concerned to the link between differential Galois theory, representation of quivers and seismic models where the square of slowness has constant hessian and there is not interaction between the positions.

According to the Hamiltonian given in (2.10), the Eikonal equation (2.8) can be seen as a subsystem of the Hamiltonian in three degrees of freedom

$$(3.1) \quad H = \frac{\|\mathbf{p}\|^2}{2} - \frac{1}{2c^2(\mathbf{q})},$$

that is, the potential energy for the Hamiltonian H is given by

$$V(\mathbf{q}) = -\frac{1}{2c^2(\mathbf{q})}, \quad V(q_1, q_2, q_3) \in \mathbb{R}(q_1, q_2, q_3) \subseteq \mathbb{C}(q_1, q_2, q_3).$$

In particular, we consider some special cases present in [20], which are summarized in the following result.

THEOREM 3.1. *For a seismic model with constant Hessian of square of slowness without interaction of positions the following statements holds:*

- (1) *The differential Galois group of the variational equation of the Hamiltonian system is isomorphic to a triple tensor product with additive and multiplicative groups of complex numbers.*
- (2) *Let Γ^2 be the quiver with two vertices (ϵ_0 and ϵ_1) and one arrow α from the vertex ϵ_1 to vertex ϵ_0 . If $\varphi_\alpha : \mathbb{C}\Gamma \rightarrow M_2(\mathbb{C})$ is the representation for the arrow α , then $\exp(t\varphi_\alpha(\alpha)) = \mathbb{G}_a$ and $\exp(t(\epsilon_0 - \epsilon_1)) = \mathbb{G}_m$.*

Proof. We proceed according to each item.

- (1) By the Equation (3.1) and the hypothesis, $Hess(1/c^2(q_1, q_2, q_3))$ is constant. Since

$$Hess\left(\frac{1}{2}(p_1^2 + p_2^2 + p_3^2)\right)$$

is constant, we get that $Hess(H)$ is also constant. On the other hand, by the Equation (2.12), we see that the coefficient matrix of the variational equation is constant and is given by

$$A = X'_H = J_3 Hess(H).$$

For instance, by the Equation (2.11), the Hamilton equations form a linear differential system given by

$$\dot{z} = J\nabla H = X_H.$$

Therefore the variational equation and the Hamiltonian system are equivalent. By the Equation (2.13) and because there is not interaction between the positions, the Hamiltonian is given by

$$H = \frac{1}{2}||\mathbf{p}||^2 + \mathbf{q}^T \text{diag}(b_1, b_2, b_3)\mathbf{q},$$

i.e., equivalently

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{2} + b_1 q_1^2 + b_2 q_2^2 + b_3 q_3^2, \quad b_i \in \mathbb{C}.$$

This Hamiltonian can be written as $H = H_1 + H_2 + H_3$, which is integrable owing to the rational first integrals H_1 , H_2 and H_3 are in involution, i.e., $\{H_1, H_2\} = \{H_1, H_3\} = \{H_2, H_3\} = 0$, and they are also independent. Since the Hamiltonian system is linear, we obtain general solutions $\gamma = \gamma(t)$ and for instance, $X'_H|_\gamma = X'_H$. For instance, the variational equation is equivalent to a linear differential system of three uncoupled second order differential equations with constant coefficients is given by

$$\ddot{\xi}_1 = b_1 \xi_1, \quad \ddot{\xi}_2 = b_2 \xi_2, \quad \ddot{\xi}_3 = b_3 \xi_3.$$

These second order differential equations have the same differential field $\mathcal{K} = \mathbb{C}$, which is the field of complex numbers. Furthermore, they have only one singularity namely, $t = \infty$, which is of irregular type due after the change of variable $t = \frac{1}{z}$, $z = 0$ is singular irregular point of

$$y'' + \frac{2}{z}y' + \frac{b_i}{z^4}y = 0.$$

By Morales-Ramis theory, the differential Galois group of the variational equation must be virtually abelian, which in our case corresponds to connected groups computed as follows. To obtain the Picard-Vessiot extensions \mathcal{L} we can consider two cases for each second order differential equation. The first case corresponds to $b_i = 0$, for some $i \in \{1, 2, 3\}$. Thus, the

basis of solutions is given by $\mathcal{B} = \langle 1, t \rangle$ and the Picard-Vessiot extension is given by $\mathcal{L} = \mathbb{C}(t)$. Thus, the differential automorphisms $\sigma : \mathcal{L} \rightarrow \mathcal{L}$ such that $\sigma|_{\mathcal{K}} = id$ are given by

$$\sigma(1) = 1, \quad \text{and} \quad \sigma\left(\frac{d}{dt}t\right) = \frac{d}{dt}t = \frac{d}{dt}\sigma(t),$$

that is, $\sigma(t) = t + \mu$. This implies that

$$\text{DGal}(\mathcal{L}/\mathbb{C}) = \{\sigma_\mu : \sigma_\mu(y) = y + \mu, \forall y \in \mathcal{B}, \forall \mu \in \mathbb{C}\} \cong \mathbb{G}_a.$$

The second case corresponds to $b_j \neq 0$, for some $j \in \{1, 2, 3\}$. Thus, the basis of solutions is given by $\mathcal{B} = \langle \exp(\sqrt{b_j}t), \exp(-\sqrt{b_j}t) \rangle$ and the Picard-Vessiot extension is given by $\mathcal{L} = \mathbb{C}(\exp(\sqrt{b_j}t))$. Thus, the differential automorphisms $\sigma : \mathcal{L} \rightarrow \mathcal{L}$ such that $\sigma|_{\mathcal{K}} = id$ are given by

$$\sigma\left(\frac{d}{dt} \ln(\exp(\sqrt{b_j}t))\right) = \frac{d}{dt} \ln(\exp(\sqrt{b_j}t)) = \frac{d}{dt} \sigma(\ln(\exp(\sqrt{b_j}t)))$$

$$\text{and} \quad \sigma(\exp(-\sqrt{b_j}t)) = \sigma^{-1} \exp(\sqrt{b_j}t),$$

that is, $\sigma(y) = \lambda y$, where $y \in \mathcal{B}$. This implies that

$$\text{DGal}(\mathcal{L}/\mathbb{C}) = \{\sigma_\lambda : \sigma_\lambda(y) = \lambda y, \forall y \in \mathcal{B}, \forall \lambda \in \mathbb{C}^*\} \cong \mathbb{G}_m.$$

Since there are three second order differential equations, there are eight possibilities for the differential Galois group of the variational equation, which is isomorphic to

$$\mathbb{G}_i \otimes \mathbb{G}_j \otimes \mathbb{G}_k, \quad i, j, k \in \{a, m\}.$$

- (2) Consider the quiver $\Gamma^2: 1 \xrightarrow{\alpha} 0$, if we also consider a \mathbb{C} -basis $\{z_0, z_1\}$ of $M[\alpha]$ and let $p = \alpha_1$, with $\alpha_1 = \alpha$, then $\varphi_\alpha(0)(z_0) = z_0$, $\varphi_\alpha(0)(z_1) = 0$, $\varphi_\alpha(1)(z_0) = 0$, $\varphi_\alpha(1)(z_1) = z_1$, $\varphi_\alpha(\alpha)(z_0) = 0$, and $\varphi_\alpha(\alpha)(z_1) = z_0$. Therefore we have that

$$\varphi_\alpha(\epsilon_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_\alpha(\epsilon_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \varphi_\alpha(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This finishes the proof of the Theorem 3.1. ■

REMARK 3.2. Theorem 3.1 provides an algebraic study of seismic models in seismology, where seismic models with constant velocity field are included in a natural way. Moreover, the square of the slowness for these models with null Hessian have linear heterogeneity in all directions, and the Hamiltonian H can be easily separated in three hamiltonians in one degree of freedom. While for a constant non-null Hessian, the square of the slowness is highly heterogeneous and it can be represented as a quadratic polynomial in three variables without interactions between the positions and the Hamiltonian is separable into three hamiltonians as in the previous case.

Final Remarks

In this paper we studied algebraically some seismic models developed in [20]. Using the Hamiltonian approach were considered seismic models with constant Hessian of the square of slowness, without interactions between the positions that are equivalent to hamiltonians of three uncoupled harmonic oscillators. Given that the Hamiltonian system is integrable trough rational first integrals, Morales-Ramis

Theorem 2.4, we obtained abelian differential Galois groups and the representation of the quiver, that allow us to obtain such abelian differential Galois groups, for the variational equations of Hamiltonian systems associated to such seismic models.

We restricted to the case of square slowness without interactions, to motivate the readers through examples about the link between seismology, differential Galois theory and representation theory of quivers. This approach can be extended to more general and complicated examples of seismic models such as anisotropic and inelastic seismic models. It is expected that, seismic models with constant Hessian of square of slowness with interactions between the positions should be studied with this approach in future developments.

Acknowledgements

The authors were supported by the COLCIENCIAS-ECOPETROL project (Contract RC. No. 0266-2013) and CODI (Estrategia de Sostenibilidad 2016-2017, Universidad de Antioquia, UdeA). The first author thanks the School of Basic and Biomedical Sciences at the Universidad Simón Bolívar for the release time to work in this project. The third author thanks the Department of Basic Science at the Universidad de Medellín for the release time to work in this project.

References

- [1] P. Acosta-Humánez, M. Álvarez-Ramírez, D. Blázquez-Sanz, and J. Delgado. Non-integrability criterium for normal variational equations around an integrable subsystem and an example: The Wilberforce spring-pendulum. *Discrete and Continuous Dynamical Systems - Series A (DCDS-A)*, 33(1):965–986, 2013.
- [2] P. Acosta-Humánez, M. Álvarez-Ramírez, and J. Delgado. Non-integrability of some few body problems in two degrees of freedom. *Qualitative Theory of Dynamical Systems*, 8(2):209–239, 2009.
- [3] P. Acosta-Humánez and D. Blázquez-Sanz. Hamiltonian systems and variational equations with polynomial coefficients. *Dynamic systems and applications*, 5(1):6–10, 2008.
- [4] P. Acosta-Humánez and D. Blázquez-Sanz. Non-integrability of some hamiltonians with rational potentials. *Discrete and Continuous Dynamical Systems Series B*, 10(2 & 3):265–293, 2008.
- [5] P. Acosta-Humánez, D. Blázquez-Sanz, and C. Vargas-Contreras. On hamiltonian potentials with quartic polynomial normal variational equations. *Nonlinear Studies. The international journal*, 16:299–313, 2009.
- [6] P. Acosta-Humánez, J.J. Morales-Ruiz, and J.-A. Weil. Galoisian approach to integrability of Schrodinger Equation. *Report on Mathematical Physics*, 67(3):305–374, 2011.
- [7] P. Acosta-Humánez and Ch. Pantazi. Darboux integrals for Schrodinger planar vector fields via Darboux transformations. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 8(043):1–26, 2012.
- [8] P. Acosta-Humánez and E. Suazo. Liouvillian propagators, Riccati equation and differential Galois theory. *J. Phys. A: Math. Theor.*, 46(45,455203):1–17, 2013.
- [9] P. Acosta-Humánez and E. Suazo. Liouvillian propagators and degenerate parametric amplification with time-dependent pump amplitude and phase. *Analysis, Modelling, Optimization, and Numerical Techniques, Springer Proceedings in Mathematics & Statistics*, 121(1):295–307, 2015.
- [10] P. B. Acosta-Humánez. *Galoisian Approach to Supersymmetric Quantum Mechanics*. Phd Thesis, Universitat Politècnica de Catalunya, available in ArXiv: 0906.3532, 2009.
- [11] P. B. Acosta-Humánez. Nonautonomous Hamiltonian Systems and Morales-Ramis Theory I. the case $\ddot{x} = f(x, t)$. *SIAM Journal on Applied Dynamical Systems*, 8(1):279–297, 2009.

- [12] P. B. Acosta-Humánez. *Galoisian Approach to Supersymmetric Quantum Mechanics. The integrability analysis of the Schrodinger equation by means of differential Galois theory*. VDM Verlag, Dr Muller, Berlin, 2010.
- [13] P. B. Acosta-Humánez, S.I. Kryuchkov, E. Suazo, and S.K. Suslov. Degenerate parametric amplification of squeezed photons: Explicit solutions, statistics, means and variances. *Journal of Nonlinear Optical Physics & Materials*, 24(2, 1550021):1–27, 2015.
- [14] P. B. Acosta-Humánez, J.T. Lázaro, J. Morales-Ruiz, and Ch. Pantazi. On the integrability of polynomial vector fields in the plane by means of Picard-Vessiot theory. *Discrete and Continuous Dynamical Systems - Series A (DCDS-A)*, 35(5):1767–1800, 2015.
- [15] V. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer Verlag, New York, USA, second edition, 1989.
- [16] I. Assem, D. D. Simson, and A. Skowroński. *Elements of the Representation Theory of Associative Algebras*. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006.
- [17] M. Auslander, I. Reiten, and S. Smalø. *Representation theory of Artin Algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1995.
- [18] N. Bleistein. *Mathematical methods for wave phenomena*. Academic Press, 1986.
- [19] N. Bleistein and R. Handelsman. *Asymptotic Expansion of Integrals*. Dover, New York, USA, second edition, 1984.
- [20] V. Cerveny. *Seismic Ray Theory*. Cambridge University Press, Cambridge, UK, 2001.
- [21] J. de la Peña. *Tame algebras and derived categories*. XV Escola de Álgebra. UFRGS, Brasil, Canela-RS, first edition, 1998.
- [22] L. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, USA, second edition, 2010.
- [23] J. Fritz. *Partial Differential Equations*, volume 1 of *Applied Mathematical Sciences*. Springer Verlag, New York, USA, fourth edition, 1982.
- [24] P. Gabriel. Unzerlegbare darteilungen i. *Manuscripta Math*, 6:71–103, 1972.
- [25] P. Gabriel. Auslander-Reiten sequences and representation-finite algebras. *Proc. ICRA II (Ottawa, Canada 1979), Lecture Notes in Math., Springer-Verlag*, 831:1–71, 1980.
- [26] W. H. Gustafson. The history of algebras and their representations. *Proc. ICRA III (Puebla, Mexico 1980), Lecture Notes in Math., Springer-Verlag*, 944:1–28, 1982.
- [27] I. Kaplansky. *An introduction to differential algebra*. Hermann, 1957.
- [28] T. Kimura. On Riemann’s equations which are solvable by quadratures. *Funkcialaj Ekvacioj*, 12(1):269–281, 1969.
- [29] J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. *J. Symbolic Computation*, 2(1):3–43, 1986.
- [30] A. Magid. *Lectures on differential Galois theory*. University Lecture Series. American Mathematical Society, Providence, RI, 1994.
- [31] J. Martinet and J.P. Ramis. Theorie de galois differentielle et resommation. *Computer Algebra and Differential Equations*, 193(1):117–214, 1989.
- [32] J. Morales-Ruiz. *Differential Galois Theory and Non-Integrability of Hamiltonian Systems*. Progress in Mathematics. Birkhauser, Basel, 1999.
- [33] J. J. Morales-Ruiz and J.-P. Ramis. Galoisian obstructions to integrability of hamiltonian systems I. *Methods Appl. Anal.*, 8(1):33–96, 2001.
- [34] J. J. Morales-Ruiz and J.-P. Ramis. Galoisian obstructions to integrability of hamiltonian systems II. *Methods Appl. Anal.*, 8(1):97–112, 2001.
- [35] J. J. Morales-Ruiz and J.-P. Ramis. Integrability of dynamical systems through differential Galois theory: a practical guide. *Differential algebra, complex analysis and orthogonal polynomials, Contemp. Math., Amer. Math. Soc.*, 509(1):143–220, 2010.
- [36] J. Rauch. *Hyperbolic Partial Differential in Geometrical Optics*, volume 133 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, USA, first edition, 2012.
- [37] I. Reiten. An introduction to the representation of Artin algebras. *Bull London Math. Soc.*, 17:209–223, 1985.
- [38] C. M. Ringel. *Tame Algebras and Integral Quadratic Forms*, volume 1099 of *Lecture Notes in Mathematics*. Springer-Verlag, 1984.

- [39] J. Schleicher, M. Tygel, and P. Hubral. *Seismic True-Amplitude Imaging*, volume 12 of *SEG Geophysical Developments*. Society of Exploration Geophysics, Tulsa, OK, USA, first edition, 2007.
- [40] M.F. Singer. An outline of differential galois theory. *Computer Algebra and Differential Equations*, 121(1):3–58, 1989.
- [41] M. van der Put and M. Singer. *Galois Theory in Linear Differential Equations*. Graduate Text in Mathematics. Springer Verlag, New York, 2003.
- [42] M. Zworski. *Semiclassical Analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, USA, first edition, 2012.

(P. Acosta-Humánez) SCHOOL OF BASIC AND BIOMEDICAL SCIENCES, UNIVERSIDAD SIMÓN BOLÍVAR, BARRANQUILLA, COLOMBIA.

E-mail address: `primitivo.acosta@unisimonbolivar.edu.co--primi@intelectual.co`

(H. Giraldo) INSTITUTE OF MATHEMATICS, UNIVERSIDAD DE ANTIOQUIA, MEDELLÍN, COLOMBIA.

E-mail address: `hernan.giraldo@udea.edu.co`

(C. Piedrahita) DEPARTMENT OF BASIC SCIENCES, UNIVERSIDAD DE MEDELLÍN, MEDELLÍN, COLOMBIA.

E-mail address: `cpiedrahita@udem.edu.co`